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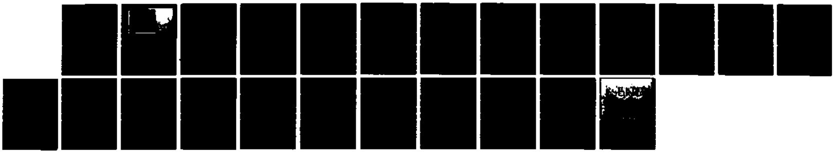

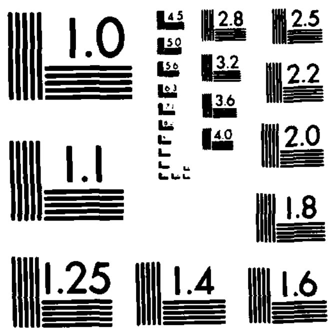

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NONLINEAR GRAVITY-CAPILLARY WAVES ON A COMPRESSIBLE VISCOUS FLUID WITH EDGE CONSTRAINTS

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May 1983

(Received April 7, 1983)

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NONLINEAR GRAVITY-CAPILLARY WAVES ON A COMPRESSIBLE VISCOUS FLUID WITH EDGE CONSTRAINTS

M. C. Shen*

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ABSTRACT

An asymptotic method is developed for the study of gravity-capillary waves in a compressible viscous fluid with edge constraints in an inclined, straight channel. The Navier-Stokes equations subject to free surface and rigid bottom conditions are reduced to a sequence of elliptic boundary problems over a cross section of the channel. Their solutions are used to determine the wave speed and to construct the Burgers equation for the evolution of the gravity-capillary waves. The Burgers equation may become ill-posed when the Reynolds number exceeds some critical value. A criterion for the stability of the flow is then defined in terms of the critical Reynolds number.

AMS (MOS) Subject Classifications: 76E30, 76N99

Key Words: Gravity-Capillary waves, compressible viscous fluid, edge constraints, Burgers equation

Work Unit Number 2 - Physical Mathematics

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and the National Science Foundation under Grant No. MCS-800-1960.

SIGNIFICANCE AND EXPLANATION

We consider a compressible viscous flow down an inclined straight channel of arbitrary cross section. The fluid is filled up to the brim of the channel so that the edges of the free surface of the fluid remain fixed. Assume that a disturbance of small amplitude and long wave length is present on the free surface; we would like to study the subsequent development of the disturbance. Within the framework of the so-called long wave approximation, we scale various variables by appropriate units and expand the solution of the Navier-Stokes equations governing the motion of a compressible viscous fluid in an asymptotic series in terms of a small parameter. The governing equations subject to prescribed boundary conditions and edge constraints are reduced to a sequence of two-dimensional linear elliptic boundary problems over a cross section of the channel. We make use of their solutions to determine the wave speed and to construct the Burgers equation for the evolution of the disturbance. The reduction of the Navier-Stokes equations subject to various boundary conditions to a single nonlinear equation, for which the solution method is well known, is our main contribution. When the Reynolds number of our problem exceeds some critical value, the Burgers equation may not be well-posed. Therefore, we use this critical Reynolds number to define a criterion for the stability of the flow.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

NONLINEAR GRAVITY-CAPILLARY WAVES ON A COMPRESSIBLE VISCOUS FLUID WITH EDGE CONSTRAINTS

M. C. Shen*

1. Introduction

In recent years there has been growing interest in the study of the initial-boundary value problem of a compressible viscous fluid. A bibliography up to 1980 may be found in the review article by Solonnikov and Kazhikov (1981). The problem of a compressible viscous fluid with heat conduction bounded by a rigid boundary was resolved by Matsumura and Nishida (1981), and that subject to free surface conditions only was considered by Tani (1981). However, it appears that the general case with both free surface and rigid boundary conditions still remains a difficult problem and it may become more untractable if the free surface and the rigid boundary intersect. Needless to say, at present no quantitative analytical results are available for these problems. On the other hand, the study of gravity-capillary waves on an inviscid fluid with edge constraints in a channel has received some attention lately (Benjamin and Scott, 1979; Benjamin, 1981; Shen 1982). It should be of great interest, therefore, to develop an asymptotic method for the investigation of the effects of compressibility, viscosity and surface tension on the propagation of surface waves subject to edge constraints in a channel. The problem we shall consider deals with a compressible viscous flow down an inclined straight channel of arbitrary cross section. We keep the edges of the free surface fixed by filling the fluid up to the brim of the channel. Suppose a surface disturbance of small amplitutde and long wave length is present in the channel. We would like to study the subsequent development of the disturbance.

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Our method of approach is based upon the so-called long wave approximation (Shih and Shen, 1975). We first introduce a small parameter as the ratio of the vertical length scale to the horizontal length scale. Then we use it to stretch various variables in the governing equations. A coordinate system is chosen to move with the disturbance at a speed near some critical speed to be determined. By expanding the solution of the governing equations in an asymptotic series, we reduce the Navier-Stokes equations with boundary conditions to a sequence of two-dimensional linear elliptic boundary value problems over a cross section of the channel. Their solutions are then used to determine the critical speed and the Burgers equation as an approximate equation for the evolution of the disturbance. The Burgers equation may become ill-posed if the Reynolds number defined in our problem exceeds some critical value. We make use of the critical Reynolds number to define a stability criterion of the disturbance. The linear stability theory of an incompressible viscous flow down an inclined plane was first studied by Yih (1963), and the ill-posed problem of the linearized Burgers equation for the same flow was considered by Carasso and Shen (1977).

We formulate our problem in Section 1. The critical speed is determined in Section 2. In Section 3, the Burgers equation is derived and the stability criterion, defined. Some special cases such as a compressible viscous flow down an inclined tube and two-dimensional flow down an inclined plane or between two planes are discussed and some remarks regarding a heat-conductive fluid and problems for further study are given in Section 4.

2. Formulation

We consider the motion of a compressible, viscous fluid in an inclined straight channel of arbitrary cross section. A rectangular coordinate system (x_1, x_2, x_3) moving with a constant speed λ^* in the x_1-direction is chosen so that the two edges of the channel lie in the x_1, x_2-plane and $H^*(x_2, x_3)$ is the equation of the channel surface (Figure 1). For simplicity, we assume that the fluid is barotropic, that is, the density ρ^* of the fluid is a given function of the pressure p^* only. In reference to the moving frame, the Navier-Stokes equations are

$$\partial \rho^*/\partial t^* + \nabla^* \cdot \rho^* \bar{q}^* = 0 \quad , \tag{1}$$

$$\rho^* D\bar{q}^*/Dt^* = -\nabla^* p^* + \rho^* \bar{g} + \mu \nabla^{*2} \bar{q}^* + (\mu/3)\nabla^*(\nabla^* \cdot \bar{q}^*) \quad , \tag{2}$$

$$\rho^* = \rho^*(p^*) \tag{3}$$

subject to the boundary conditions:

at the free surface $\quad \xi = -x_3 + \eta^*(x_1, x_2, t^*) = 0 \quad ,$

$$\sigma_{ij} n_j = T(R_1^{*-1} + R_2^{*-1}) n_i \quad , \tag{4}$$

$$D\xi/Dt^* = 0 \quad ; \tag{5}$$

at the channel wall $\quad H^*(x_2, x_3) = 0 \quad ,$

$$\bar{q} = (-\lambda^*, 0, 0) \quad ; \tag{6}$$

at the edges $\quad x_2 = \pm \ell^*, \; x_3 = 0 \quad ,$

$$\eta^* = 0 \quad . \tag{7}$$

Here $\nabla^* = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $D/Dt^* = \partial/\partial t^* + \bar{q} \cdot \nabla^*$ $\bar{q}^* = (q_1, q_2, q_3)$ is the velocity vector, $\bar{g} = (g \sin \theta, 0, -g \cos \theta)$ is the constant gravitational acceleration, θ is the inclination angle and $0 < \theta < \pi/2$, μ is the constant viscosity coefficient, σ_{ij} is the stress tensor given by

$$\sigma_{ij} = (-p^* - 2\mu \nabla^* \cdot \bar{q}^*/3)\delta_{ij} + \mu(2q_i/\partial x_j + 2q_j/\partial x_i) \quad , \tag{8}$$

$\bar{n} = (\partial\eta^*/\partial x_1, \partial\eta^*/\partial x_2, -1)$ is the normal vector to the free surface $\xi = 0$, T is the constant surface tension coefficient, and

$$R_1^{*-1} + R_2^{*-1} = [\eta^*_{x_1x_1}(1 + \eta^{*2}_{x_2}) + \eta^*_{x_2x_2}(1 + \eta^{*2}_{x_1}) - 2\eta^*_{x_1x_2}\eta^*_{x_1}\eta^*_{x_2}](1 + \eta^{*2}_{x_1} + \eta^{*2}_{x_2})^{-3/2} \quad , \tag{9}$$

where the subscript x_i denotes the partial differentiation with respect to x_i. Furthermore, we also assume that $\rho^*(p^*) > 0$, $d\rho^*/dp^* > 0$.

We introduce the following nondimensional variables

$$t = \epsilon^2 t^*/(h/g)^{1/2}, \ (x,y,z) = (\epsilon x_1, x_2, x_3)/h \quad ,$$

$$\bar{q} = (u,v,w) = (q_1, \epsilon^{-1}q_2, \epsilon^{-1}q_3)/(gh)^{1/2}, \ \rho = \rho^*/\Delta \quad ,$$

$$p = p^*/(\Delta gh), \ \eta = \eta^*/h, \ H(y,z) = H^*(y^*,z^*) \quad ,$$

$$R = \Delta(gh)^{1/2}h/\mu, \ S = T/[\mu(gh)^{1/2}], \ \lambda = \lambda^*/(gh)^{1/2} \quad ,$$

$$\ell = \ell^*/h, \ \epsilon = h/L << 1 \quad ,$$

where h and L are respectively the vertical and horizontal length scales, Δ is a reference value for the density to be chosen later. In terms of them, the governing equations and boundary conditions (1) to (7) become

$$\epsilon\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0 \quad , \tag{10}$$

$$\rho\epsilon(\epsilon u_t + uu_x + vu_y + wu_z) = -\epsilon p_x + \rho \sin\theta + R^{-1}(\epsilon^2 u_{xx} + u_{yy} + u_{zz}) + R^{-1}(\epsilon^2/3)(\nabla\cdot\bar{q})_x \tag{11}$$

$$\rho\epsilon^2(v_t + uv_x + uv_y + wv_y) = -py + R^{-1}\epsilon(\epsilon^2 v_{xx} + v_{yy} + v_{zz}) + R^{-1}(\epsilon/3)(\nabla\cdot\bar{q})_y \tag{12}$$

$$\rho\epsilon^2(w_t + uw_x + vw_y + ww_z) = -p_z - \rho\cos\theta + R^{-1}\epsilon(\epsilon^2 w_{xx} + w_{yy} + w_{zz}) + R^{-1}(\epsilon/3)(\nabla\cdot\bar{q})_z \tag{13}$$

$$\rho = \rho(p) \quad , \tag{14}$$

subject to the boundary conditions: At the free surface $z = \eta(x,y,t,\epsilon)$,

$$\epsilon[Rp + (2\epsilon/3)\nabla\cdot\bar{q} - 2\epsilon u_x]\eta_x - (u_y + \epsilon^2 v_x)\eta_y$$
$$+ u_z + \epsilon^2 w_x = -S(R_1^{-1} + R_2^{-1})\epsilon\eta_x \quad , \tag{15}$$

$$[Rp + (2\epsilon/3)\nabla\cdot\bar{q} - 2\epsilon v_y]\eta_y - \epsilon(\epsilon^2 v_x + u_y)\eta_x$$
$$+ \epsilon(v_z + w_y) = -S(R_1^{-1} + R_2^{-1})\eta_y \quad , \tag{16}$$

$$Rp + (2\epsilon/3)\nabla\cdot\bar{q} - 2\epsilon w_z + \epsilon(\epsilon^2 w_x + u_z)\eta_x + \epsilon(w_y + v_z)\eta_y$$
$$= -S(R_1^{-1} + R_2^{-1}) \quad , \tag{17}$$

where $\nabla\cdot\bar{q} = (u_x + v_y + w_z)$,

$$R_1^{-1} + R_2^{-1} = [\epsilon^2\eta_{xx}(1 + \eta_y^2) + \eta_{yy}(1 + \epsilon^2\eta_x^2) - 2\epsilon^2\eta_{xy}\eta_x\eta_y](1 + \epsilon^2\eta_x^2 + \eta_y^2)^{-3/2} \quad ,$$

$$\epsilon\eta_t + u\eta_x + v\eta_y - w = 0 \quad ; \tag{18}$$

at $H(y,z) = 0$,

$$u = -\lambda, \; v = w = 0 \quad ; \tag{19}$$

at $y = \pm\ell$, $z = 0$, $\qquad \eta = 0 \quad . \tag{20}$

3. Critical speed

Assume that the solutions $\bar{q}$, ρ, p, η and λ possess an asymptotic expansion of the form

$$\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots\cdots \quad . \tag{21}$$

We call λ_0 the critical speed. Without loss of generality, assume $\lambda = \lambda_0 + \epsilon\lambda_1$. Substitution of (21) in (10) to (20) will yield a sequence of equations and boundary conditions for the successive approximations. The equations for the zeroth approximation are the following:

$$R^{-1}\nabla^2 u_0 = -\rho_0 \sin\theta \quad \text{in} \quad \Gamma \quad , \tag{22}$$

$$u_{0z} = 0 \quad \text{on} \quad L_0 \quad , \tag{23}$$

$$u_0 = -\lambda_0 \quad \text{on} \quad L_1 \quad , \tag{24}$$

$$p_{0z} = -\rho_0 \cos\theta \quad \text{in} \quad D \quad , \tag{25}$$

$$p_{0y} = 0 \quad \text{in} \quad D \quad , \tag{26}$$

$$p_0 = 0 \quad \text{on} \quad L_0 \quad , \tag{27}$$

$$\rho_0 = \rho(p_0) \quad \text{in} \quad D \quad , \tag{28}$$

where we assume $\eta_0 = v_0 = w_0 = 0$, $\nabla^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$, u_0 is a function of y, z only, D is a cross section of the channel, L_1 is the channel boundary and L_0 is the line segment $-\ell \leq y \leq \ell$, $z = 0$, as part of the boundary of D (Figure 1). Furthermore we choose Δ so that $\rho_0(0) = 1$. From (25) to (28), it is obtained that p_0 satisfies

$$\int_0^{p_0} \rho^{-1}(p_0)dp_0 = -z \cos\theta \quad . \tag{29}$$

In turn, ρ_0 is determined from (28). Let

$$u_0 = R\phi_0 - \lambda_0 \quad . \tag{30}$$

Then by (22) to (24), ϕ_0 is the solution of

$$\nabla^2\phi_0 = -\rho_0 \sin\theta \quad \text{in} \quad D \quad , \tag{31}$$

$$\phi_{0z} = 0 \quad \text{on} \quad L_0 \quad , \tag{32}$$

$$\phi_0 = 0 \quad \text{on} \quad L_1 \quad . \tag{33}$$

The equations for the first approximation are

$$(\rho_0 u_0)_x + (\rho_1 u_0)_x + (\rho_0 v_1)_y + (\rho_0 w_1)_z = 0 \quad , \tag{34}$$

$$R^{-1}\nabla^2 u_1 = -\rho_1 \sin\theta \quad , \tag{35}$$

$$p_{1z} = -\rho_1 \cos\theta, \; p_{1y} = 0 \quad , \tag{36}$$

$$\rho_1 = \rho'(p_0)p_1 \quad ; \tag{37}$$

at $z = 0$,

$$-u_{0y}\eta_{1y} + u_{1z} + u_{0zz}\eta_1 = 0 \quad , \tag{38}$$

$$Rp_{0z}\eta_1 + Rp_1 = -S\,\eta_{yy} \quad , \tag{39}$$

$$u_0\eta_{1x} - w_1 = 0 \quad ; \tag{40}$$

at $H(y,z) = 0$,

$$u_1 = -\lambda_1, \; v_1 = w_1 = 0 \quad ; \tag{41}$$

at $y = \pm\ell$, $z = 0$, $\quad \eta_1 = 0 \quad . \tag{42}$

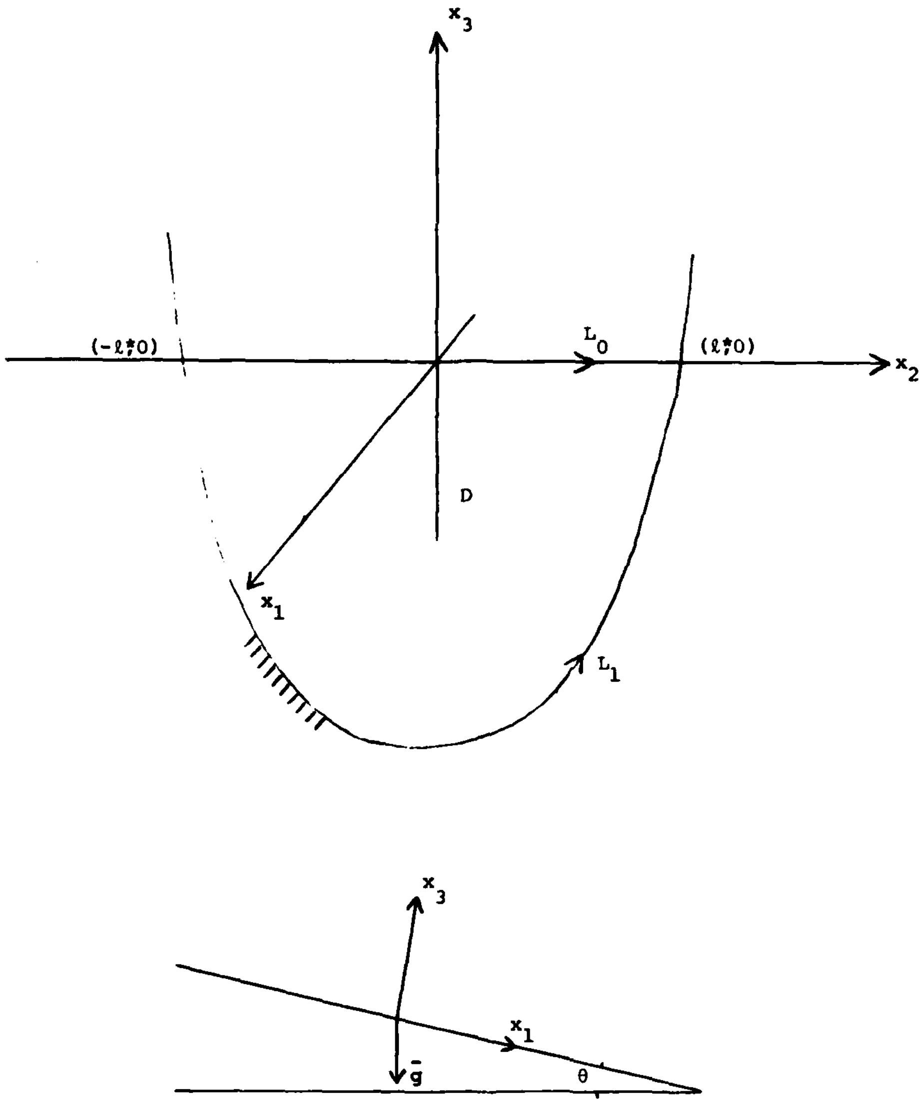

Figure 1. A cross section of the Inclined Channel

We obtain from (36) and (37) that

$$p_{1z}/p_1 = -\rho'(p_0)\cos\theta \quad .$$

By integration and making use of (25), we have

$$p_1 = \rho(p_0) f_1(x,t) \quad , \tag{43}$$

where $f_1(x,t)$, a function of x, t only, is to be determined. Next from (25), (39), (42) and (43) it follows that

$$S\eta_{1yy} - R\cos\theta\,\eta_1 = -Rf_1(x,t)$$

$$\eta_1 = 0 \quad \text{at} \quad y = \pm\ell \quad .$$

It is easy to obtain that

$$\eta_1 = \nu_1(y) f_1(x,t) \quad , \tag{44}$$

where

$$\nu_1(y) = (\cos\theta)^{-1}[1 - (\cosh k\ell)^{-1}\cosh ky] \quad ,$$

$$k = (R\cos\theta/S)^{1/2} = (\rho g h^2 \cos\theta/T)^{1/2} \quad . \tag{45}$$

Let

$$u_1 = R\phi_1(y,z) f_1 - \lambda_1 \quad . \tag{46}$$

Then from (35), (38) and (41), ϕ_1 is the solution of the following problem:

$$\nabla^2\phi_1 = -\rho'(p_0)\rho_0 \sin\theta \quad \text{in} \quad D \quad , \tag{47}$$

$$\phi_{1z} = \phi_{0y}\nu_1' - \phi_{0zz}\nu_1 \quad \text{on} \quad L_0 \quad , \tag{48}$$

$$\phi_1 = 0 \quad \text{on} \quad L_{-1} \quad . \tag{49}$$

To determine λ_0, we integrate (34) over D, use the divergence theorem, (30), (37), (40), (41), (43) and (44) to obtain

$$\int_{L_0}[R\phi_0 - \lambda_0]\nu_1 f_{1x}\,dy = -\int_D\!\!\int[\rho_0 R\phi_1 + \rho'(p_0)\rho_0(Rp_0 \cdot \lambda_0)]f_{1x}\,dA \quad ,$$

and

$$\lambda_0 = R\Big[\int_D\!\!\int \rho'(p_0)\rho_0\,dA + \int_{L_0}\nu_1\,dy\Big]^{-1}\Big[\int_D\!\!\int(\rho_0\phi_1 + \rho'(p_0)\rho_0\phi_0)\,dA + \int_{L_0}\nu_1\phi_0\,dy\Big] \quad , \tag{50}$$

which yields the expression for the critical speed. We write

$$\lambda_0 = R\lambda_0^* \quad , \tag{51}$$

where λ_0^* is independent of R.

4. Burgers equations

Our next step is to derive the Burgers equation for $f_1(x,t)$, and we proceed to the equations for the second approximation:

$$\rho_{1t} + (\rho_0 u_2 + \rho_1 u_1 + \rho_2 u_0)_x + (\rho_0 v_2 + \rho_1 v_1)_y + (\rho_0 w_2 + \rho_1 w_1)_z = 0 \quad , \tag{52}$$

$$\rho_0(u_0 u_{1x} + v_1 u_{0y} + w_1 u_{0z}) = -p_{1x} + \rho_2 \sin\theta + R^{-1}\nabla^2 u_2 \quad , \tag{53}$$

$$p_{2y} = R^{-1}\nabla^2 v_1 + (1/3)R^{-1}(\nabla\cdot\bar{q})_y \quad , \tag{54}$$

$$p_{2z} = -\rho_2 \cos\theta + R^{-1}\nabla^2 w_1 + (1/3)R^{-1}(\nabla\cdot\bar{q}_2)_z \quad , \tag{55}$$

$$\rho_2 = \rho'(p_0)p_2 + \rho''(p_0)p_1^2/2 \quad ; \tag{56}$$

at $z = 0$,

$$-u_{0y}\eta_{2y} - u_{1y}\eta_{1y} + u_{0zz}\eta_2 + u_{0zz}\eta_1^2/2 + u_{1zz}\eta_1 + u_{2z} = 0 \quad , \tag{57}$$

$$-u_{0y}\eta_{1x} + v_{1z} + w_{1y} = 0 \quad , \tag{58}$$

$$Rp_2 + Rp_{0z}\eta_2 + Rp_{1z}\eta_1 + (2/3)\nabla\cdot\bar{q}_1 - 2w_{1z} = -S\eta_{2yy} \quad , \tag{59}$$

$$\eta_{1t} + u_1\eta_{1x} + u_0\eta_{2x} + v_1\eta_{1y} = w_2 + w_{1z}\eta_1 \quad ; \tag{60}$$

at $H(y,z) = 0$,

$$u_2 = v_2 = w_2 = 0 \quad ; \tag{61}$$

at $y = \pm\ell$, $z = 0$, $\quad \eta_2 = 0 \quad .$ (62)

We first go back to (34) and make use of (37), (43) and (46) to express (34) as

$$\begin{aligned}(\rho_0 v_1)_y + (\rho_0 w_1)_z &= -(\rho_0 u_1 + \rho_1 u_0)_x \\ &= -R[\rho_0\phi_1 + (\phi_0 - \lambda_0^*)\rho'(p_0)\rho_0]f_{1x}\end{aligned} \tag{63}$$

Then we construct a function $Q(y,z)$ as a solution of the following equations:

$$\nabla\cdot(\rho_0\nabla Q_1) = -\rho_0\phi_1 + (\phi_0 - \lambda_0^*)\rho'(p_0)\rho_0 \quad \text{in} \quad D \quad , \tag{64}$$

$$Q_{1z} = (p_0 - \lambda_0^*)\nu_1 \quad \text{on} \quad L_0 \quad , \tag{65}$$

$$Q_n = 0 \quad \text{on} \quad L_1 \quad . \tag{66}$$

The Neumann problem posed by (64) to (66) is solvable because of (50). From (63), (64), we have

$$(\rho_0 v_1 - \rho_0 RQ_{1y}f_{1x})_y + (\rho_0 w_1 - \rho_0 RQ_{1z}f_{1x})_z = 0 \quad . \tag{67}$$

It follows from (67) that we may define a function $R\Phi_1(x,y)$ such that

$$R\Phi_{1z}f_{1x} = \rho_0 v_1 - \rho_0 RQ_{1y}f_{1x} \quad ,$$

$$- R\Phi_{1y}f_{1x} = \rho_0 w_1 - \rho_0 RQ_{1z}f_{1x} \quad .$$

Hence,

$$v_1 = R(\rho_0^{-1}\Phi_{1z} + Q_{1y})f_{1x} \quad , \tag{68}$$

$$w_1 = R(-\rho^{-1}\Phi_{1y} + Q_{1z})f_{1x} \quad . \tag{69}$$

We now cross-differentiate (54) and (55) and subtract one equation from another to obtain

$$R^{-1}\nabla^2(v_{1z} - w_{1y}) = -\rho_{2y} \cos\theta \quad .$$

By substituting (68) and (69) for v_1 and w_1 in the above equation and making use of (46), (54), (68) and (69), it is obtained that

$$\nabla^2[\nabla\cdot(\rho_0^{-1}\Phi_1)] = -\cos\theta\rho'(p_0)[\nabla^2(\rho_0^{-1}\Phi_{1z}) + \nabla^2 Q_{1y} + (1/3)\phi_1$$
$$+ (\rho_0^{-1}\Phi_{1z})y + (\rho_0^{-1}\Phi_{1y})z + \nabla^2 Q_1] \quad \text{in} \quad D \quad . \tag{70}$$

The boundary conditions for (70) can be derived from (40), (41), and (58). It follows from (30), (44), (68) and (69) that

$$\left.\begin{aligned} &\Phi_{1y} = 0, \qquad (\rho_0^{-1}\Phi_{1z})z = \phi_{0y}v_1 - 2[(\phi_0 - \lambda_0^*)v_1]y \quad \text{on} \quad L_0 \quad , \\ &\rho_0^{-1}\Phi_{1z} + Q_{1y} = 0, \quad \rho_0^{-1}\Phi_{1y} - Q_{1z} = 0 \quad \text{on} \quad L_1 \quad . \end{aligned}\right\} \tag{70}$$

Suppose Q_1 and Φ_1 have been found, and in the following we shall successively construct manageable expressions for p_2, η_2 and u_2. We multiply both (54) and (55) by ρ_0^{-1} and make use of (56) to obtain

$$(\rho_0^{-1}p_2)y = \rho_0^{-1}R^{-1}[\nabla^2 v_1 + (1/3)(\nabla\cdot\bar{q}_1)y] \quad , \tag{72}$$

$$(\rho_0^{-1}p_2)_z = \rho_0^{-1}[-\rho''(p_0)p_1^2\cos\theta/2 + R^{-1}\nabla^2 w_1 + (1/3)R^{-1}(\nabla\cdot\bar{q}_1)_z] \quad . \tag{73}$$

In D, a fixed point (x, y_0, z_0) and a smooth curve Γ from (x_0, y_0, z_0) to (x_0, y, z) are chosen. Then from (72) and (73), we obtain

$$p_2 = \rho_0 \int_2 \rho_0^{-1}R^{-1}[\nabla^2 v_1 + (1/3)(\nabla\cdot\bar{q}_1)y]dy$$
$$+ \rho_0^{-1}[-\rho''(p_0)p_1^2\cos\theta/2 + R^{-1}\nabla^2 w_1 + (1/3)R^{-1}(\nabla\cdot\bar{q})z]dz + \rho_0 f_\alpha(x,t) \tag{74}$$
$$= p_{20}f_1^2 + p_{21}f_{1x} + \rho_0 f_2 \quad ,$$

where f_2 is an unknown function of x and t, and by (43), (46), (68) and (69),

$$p_{20} = -(\rho_0/2) \int_L \rho''(p_0)\rho_0^2 \cos\theta \, dz \quad ,$$

$$p_{21} = \rho_0 \int_L \rho_0^{-1}\{\nabla^2(\rho_0^{-1}\phi_{1z} + \Omega_{1y}) + (1/3)[\phi_{1y} + (\rho_0^{-1}\phi_{1z} + \Omega_{1y})_{yy}]$$

$$+ (-\rho_0^{-1}\phi_{1y} + \Omega_{1z})_{zy}\}dy + \{\nabla^2(\rho_0^{-1}\phi_{1y} + \Omega_{1z}) + (1/3)[\phi_{1z}$$

$$+ (\rho_0^{-1}\phi_{1z} + \Omega_{1y})_{yz} + (-\rho_0^{-1}\phi_{1y} + \Omega_{1z})_{zz}]\}dz \quad .$$

Next we turn to the equation (59) for η_2. By (25), (36), (37), (43), (44), (46), (62), (68), and (69), we may express (59) as

$$\eta_{2yy} - (R/S)\cos\theta\,\eta_2 = (R/S)[-p_{20} + \rho'(0)\cos\theta]f_1^2$$

$$- (R/S)\{p_{21} + (2/3)[\phi_1 + (\rho_0^{-1}\phi_{1y} + \Omega_{1y})_y] - (4/3)(-\rho_0^{-1}\phi_{1y} + \Omega_{1z})_z\}f_{1x}$$

$$\eta_2 = 0 \quad \text{at} \quad y = \pm\ell \quad .$$

Let $G(y,z)$ be the Green's function for $(d^2/dy^2 - k^2)\eta_2$ with $\eta_2 = 0$ at $y = \pm\ell$, it is easily found that

$$G(y,z) = \sinh k(\zeta-\ell)\sin k(y+\ell)/(k \sinh 2k\ell), \ -\ell < y < \zeta \quad ,$$

$$\sinh k(\zeta+\ell)\sinh k(y-\ell)/(k \sinh 2k\ell), \ \zeta < y < \ell \quad .$$

Then

$$\eta_2 = \nu_{20}f_1^2 + \nu_{21}f_{1x} + \nu_1 f_2 \quad , \tag{75}$$

where

$$\nu_{20} = \int_{-\ell}^{\ell} G(y,\zeta)(R/S)[-p_{20} + \rho'(0)\cos\theta]_{z=0}dy \quad ,$$

$$\nu_{21} = \int_{-\ell}^{\ell} G(y,\zeta)(R/S)\{p_{21} + (2/3)[\phi_1 + (\rho_0^{-1}\phi_{1z} + \Omega_{1y})_y]$$

$$- (4/3)(-\rho_0^{-1}\phi_{1y} + \Omega_{1z})_z\}_{z=0}dy \quad .$$

Finally we take up u_2. Making use of the expressions derived before for u_0, $\bar{q}_1$, p_1, p_2, η_1 and η_2, we obtain from (53), (56), (57) and (61) that

$$R^{-1}\nabla^2 u_2 = \{R^2\rho_0[(\phi_0-\lambda_0^*)\phi_1 + (\rho_0^{-1}\phi_{1z} + \Omega_{1y})\phi_{0y} + (-\rho_0^{-1}\phi_{1y} + \Omega_{1z})\phi_{0z}]$$

$$+ \rho_0 - p_{21}\sin\theta\,\rho'(p_0)\}f_{1x} - [\rho'(p_0)p_{20} + \rho''(p_0)\rho_0/2]\mathrm{sinc}\,\theta\, f_1^2$$

$$- \rho_0 \sin\theta\, f_2 \quad \text{in } D \ ,$$

$$u_{2z} = R(\phi_{0y}\nu_{21}' - \phi_{0zz}\nu_{21})f_{1x} + R(\phi_{0y}\nu_{20}' + \phi_{1y}\nu_1' - \phi_{0zz}\nu_{20}$$

$$- \phi_{0zzz}\nu_1^2/2 - \phi_{1zz}\nu_1)f^2 + R(\phi_{0y}\nu_1' - \phi_{0zz}\nu_1)f_2 \quad \text{on } L_0 \ ,$$

$$u_2 = 0 \quad \text{on } L_1 \ .$$

As observed from the above equations, we may express u_2 in terms of f_{1x}, f^2 and f_2 as follows:

$$u_2 = [R^3\phi_{23}(y,z) + R\phi_{21}(y,z)]f_{1x} + R\phi_3(y,z)f_1^2 + R\phi_1 f_2 \ . \tag{76}$$

Here ϕ_2, ϕ_3, ϕ_4 are the solutions of the following problems:

$$\nabla^2\phi_{23} = \rho_0[(\phi_0 - \lambda_0^*)\phi_1 + (\rho_0^{-1}\phi_{1z} + \Omega_{1y})\phi_{0y} + (-\rho_0^{-1}\phi_{1y} + \Omega_{1z})\phi_{0z}], \quad \text{in } D \ ,$$

$$\phi_{23z} = 0 \quad \text{on } L_0 \ ,$$

$$\phi_{23} = 0 \quad \text{on } L_1 \ ;$$

$$\nabla^2\phi_{21} = \rho_0 - p_{21}\rho'(p_0)\sin\theta \quad \text{in } D \ ,$$

$$\phi_{21z} = \phi_{0y}\nu_{21}' - \phi_{0zz}\nu_{21} \quad \text{on } L_1 \ ,$$

$$\phi_{21} = 0 \quad \text{on } L_0 \ ;$$

$$\nabla^2\phi_3 = -[\rho'(p_0)p_{20} + \rho''(p_0)\rho_0/2]\sin\theta \quad \text{in } D \ ,$$

$$\phi_{3z} = \phi_{0y}\nu_{20}' + \phi_{1y}\nu_1' - \phi_{0zz}\nu_{20} - \phi_{0zzz}\nu_1^2/2 - \phi_{1zz}\nu_1 \quad \text{on } L_0 \ ,$$

$$\phi_3 = 0 \quad \text{on } L_1 \ .$$

Now we are in a position to derive the Burgers equation for f_1. As before for the derivation of the critical speed, we integrate (52) over D and apply the divergence theorem and (61) to obtain

$$\iint_D [(\rho_0 v_2)_y + (\rho_0 w_2)_z] dA + \int_{L_0} w_2 dy$$
$$= -\iint_D \rho_{1t} + (\rho_0 u_2 + \rho_1 u_1 + \rho_2 u_0)_x dA - \int_{L_0} \rho_1 w_1 dy \quad , \tag{77}$$

where we note that $\rho_0 = 1$ on L_0. By making use of (30), (37), (40), (43), (44), (51), (56), (60), (74), (75) and (76), and rearranging the terms, (77) becomes

$$m_0 f_{1t} + m_1 f_{1x} + m_2 f f_{1x} = m_3 f_{1xx} \quad . \tag{78}$$

Here the coefficient of f_2 vanishes because of (50), and

$$m_0 = \int_{L_0} \nu_1 dy + \iint_D \rho'(p_0)\rho_0 dA \quad , \tag{79}$$

$$m_1 = -\lambda_1 m_0 \quad , \tag{80}$$

$$m_2 = R\int_{L_0} [2(\phi_0 - \lambda_0^*)\nu_{20} + \nu'(\rho_0^{-1}\phi_{1z} + \varrho_{1y}) - \nu_1(-\rho_0^{-1}\phi_{1y} + \varrho_{1z})_z$$
$$+ \rho'(p_0)(\phi_0 - \lambda_0^*)\nu_1] dy + \iint_D [2\rho_0\phi_3 + 2\rho''(p_0)\rho_0\phi_1$$
$$+ 2\rho'(p_0)(\phi_0 - \lambda_0^*)p_{20} + \rho''(p_0)\rho_0^2(\phi_0 - \lambda_0^*)] dA \quad , \tag{81}$$

$$m_3 = -\iint_D [\rho_0(R^3\phi_{23} + R\phi_{21}) + R\rho'(p_0)p_{21}(\phi_0 - \lambda_0^*)] dA$$
$$- \int_{L_0} R(\phi_0 - \lambda_0^*)\nu_{21} dy \quad . \tag{82}$$

(78) with (79) to (82) is our main result. It is well known that the Burgers equation is not well posed if m_3 becomes negative. We set $m_3 = 0$ and solve for R to obtain

R_c = critical Reynolds number

$$= \{[-\iint_D \rho_0\phi_{23} dA]^{-1} [\iint_D [\rho_0\phi_{21} + \rho'(p_0)p_{21}(\phi_0 - \lambda_0^*)] dA + \int_{L_0} (p_0 - \lambda_0^*)\nu_{21} dy]\}^{1/2} .$$

Assume R_c is positive and finite and $m_3 < 0$ when $R > R_c$. Then the Burgers equation is ill-posed for $R > R_c$ and we define $R < R_c$ as a criterion for the stability of the flow.

5. Discussion

In the following we consider some special cases, which can be easily dealt with by our previous results.

(1) Fixed boundary case

This case corresponds to a compressible viscous flow down an inclined tube of uniform cross section. We only have the boundary condition $u = -\lambda$, $v = w = 0$ on $h(y,z) = 0$. In all the elliptical boundary value problems, we extend the boundary condition on L_1 to the whole boundary. Evidently in (79) to (82) all the line integrals on L_0 should be dropped.

(2) Two-dimensional case

In this case, the solution of the governing equations (1) to (6) is independent of the coordinate y, and we also assume $v \equiv 0$. Furthermore, the effect of surface tension will not manifest itself up to the equations for the second approximation. First we consider a compressible viscous fluid flow down an inclined plane. p_0 is still determined by (29) and ρ_0, by (28). (31) to (33) become

$$\phi_0'' = -\rho_0 \sin\theta \quad \text{in} \quad -1 < z < 0 \ ,$$

$$\phi_0' = 0 \qquad \text{at} \quad z = 0 \ ,$$

$$\phi_0 = 0 \qquad \text{at} \quad z = -1 \ ,$$

where ϕ_0 is a function of z only. For the first approximation, p_1 is given by (43). (44) should be replaced by

$$\eta_1 = f_1 \sec\theta \ . \tag{83}$$

Noting that $\eta_{1y} = 0$ in (38), (47) to (49) now become

$$\phi_1'' = -\rho'(p_0)\rho_0 \sin\theta \quad \text{in} \quad -1 < z < 0 \ ,$$

$$\phi_1' = -\phi_0'' \sec\theta \qquad \text{at} \quad z = 0 \ ,$$

$$\phi_1 = 0 \qquad \text{at} \quad z = -1 \ .$$

We integrate (34) without $(\rho_0 v_1)_y$ with respect to z from -1 to z to obtain

$$\begin{aligned} w_1 &= -\rho_0^{-1}\int_{-1}^{z}(\rho_0 u_1 + \rho_1 u_0)_x dz \\ &= RF(z)f_{1x} \end{aligned} \tag{84}$$

where

$$F(z) = -\rho_0^{-1} \int_{-1}^{z} [\rho_0\phi_1 + \rho'(p_0)\rho_0(\phi_0 - \lambda_0 R^{-1}]dz \quad .$$

At $z = 0$, (40) holds and it follows from (83) and (84) that

$$\lambda_0 = R\lambda_0^* \quad , \tag{85}$$

where

$$\lambda_0^* = (\int_{-1}^{0} \rho'(p_0)\rho_0 dz + \sec\theta)^{-1}[\int_{-1}^{0}(\rho_0\phi_1 + \rho'(p_0)\rho_0\phi_0)dz + \phi_0(0)\sec\theta] \quad .$$

Since the expression for w_1 is known, we need not consider the equations (63) to (71). From (74) we obtain

$$\begin{aligned} p_2 &= \rho_0\int_{-1}^{z} \rho_0^{-1}[-\rho''(p_0)p_1^2 \cos\theta/2 + R^{-1}w_{1zz} + (1/3)R^{-1}(u_{1x} + w_{1z})_z]dz \\ &\quad + \rho_0 f_2(x,t) \\ &= p_{20}f_1^2 + p_{21}f_{1x} + \rho_0 f_2 \end{aligned} \tag{86}$$

where

$$p_{20} = -(\rho_0/2) \int_{-1}^{z} \rho''(p_0)\rho_0^2 \cos\theta \, dz \quad ,$$

$$p_{21} = \rho_0 \int_{-1}^{z} \rho_0^{-1}[(4/3)F'' + (1/3)\phi_1']dz \quad .$$

With similar changes in (59), we have

$$\eta_2 = \nu_{20}f_1^2 + \nu_{21}f_{1x} + (\sec\theta)f_2 \quad , \tag{87}$$

where

$$\nu_{20} = [p_{20}(0) - \rho'(0)]\sec\theta \quad ,$$

$$\nu_{21} = [p_{21}(0) + (2/3)\phi_1(0) - (4/3)F'(0)]\sec\theta \quad .$$

Finally, as in the three dimensional case, we express u_2 as

$$u_2 = [R^3\phi_{23}(z) + R\phi_{21}(z)]f_{1x} + R\phi_3(z)f_1^2 + R\phi_1 f_2 \quad , \tag{88}$$

when

$$\phi_{23}'' = \rho_0[(\phi_0-\lambda_0^*)\phi_1 + F\phi_0'] \quad \text{in} \quad -1 < z < 1 \quad ,$$

$$\phi_{23}' = 0 \quad \text{at} \quad z = 0 \quad ,$$

$$\phi_{23} = 0 \quad \text{at} \quad z = -1 \quad ;$$

$$\phi_{21}'' = \rho_0 - p_{21}\rho'(p_0)\sin\theta \quad \text{in} \quad -1 < z < 1 \ ,$$

$$\phi_{21}' = -\phi_0''\nu_{21} \quad \text{at} \quad z = 0 \ ,$$

$$\phi_{21} = 0 \quad \text{at} \quad z = -1 \ ;$$

$$\phi_3'' = -[\rho'(p_0)p_{20} + \rho''(p_0)\rho_0^2/2]\sin\theta \quad \text{in} \quad -1 < z < 0 \ ,$$

$$\phi_3' = -\phi_0''\nu_{20} - \phi_0''' \sec^2\theta/2 - \phi_1'' \sec\theta \quad \text{at} \quad z = 0 \ ,$$

$$\phi_3 = 0 \quad \text{at} \quad z = -1 \ .$$

To derive the Burgers equation, we integrate (52) without $(\rho_0 v_2 + \rho_1 v_1)_y$ with respect to z from -1 to 0 and obtain

$$m_0 f_{1t} + m_1 f_{1x} + m_2 f_1 f_{1x} = m_3 f_{1xx} \ ,$$

where

$$m_0 = \sec\theta + \int_{-1}^{0} \rho'(p_0)\rho_0 dz$$

$$= \sec\theta + [\rho_0(-1) - 1]\sec\theta$$

$$= \rho_0(-1)\sec\theta \ ,$$

$$m_1 = -\lambda_1 m_0 \ ,$$

$$m_2 = 2R[\phi_0(0) - \lambda_0^*]\nu_{20} - RF'(0)\sec\theta$$

$$+ \int_{-1}^{0}[2\rho_0 R\phi_3 + 2R(p_- - \lambda_0^*)\rho'(p_0)p_{20} + 2\rho'(p_0)\rho_0 + \rho''(p_0)\rho_0^2]dz \ ,$$

$$m_3 = -R[\phi_0(0) - \lambda_0^*]\nu_{21} - \int_{-1}^{0}[\rho_0(R^3\phi_{23} + R\phi_{21}) + R\rho'(p_0)(\phi_0 - \lambda_0^*)p_{21}]dz \ .$$

As before, we set $m_3 = 0$ and solve R to obtain

$$R_c = \{[-\int_{-1}^{0}\rho_0\phi_{23}dz]^{-1}[\int_{-1}^{0}[\rho_0\phi_{21} + \rho'(p_0)p_{21}(\phi_0 - \lambda_0^*)]dz$$

$$+ (\phi_0(0) - \lambda_0^*)\nu_{21}]\}^{1/2} \ .$$

Assume R_c is positive and finite. If for $R > R_c$, m_3 is negative, then we define $R < R_c$ as the criterion for the stability of the flow.

Suppose the plane $z = 0$ is a rigid boundary. Then in all the two-point boundary value problems, we simply impose at $z = -1$ the same boundary condition as at $z = 0$. The coefficients in the Burgers equation become

$$m_0 = [\rho_0(-1) - 1] \sec\theta \ ,$$

$$m_1 = -\lambda_1 m_0 \ ,$$

$$m_2 = \int_{-1}^{0} [2\rho R\phi_3 + 2R(\phi_0 - \lambda_0^*)\rho'(p_0)p_{20} + 2\rho'(p_0)\rho_0 + \rho''(p_0)\rho_0^2]dz \quad ,$$

$$m_3 = -\int_{-1}^{0} [\rho_0(R^3\phi_{23} + R\phi_{21}) + r\rho'(p_0)(\phi_0 - \lambda_0^*)p_{21}]dz \quad .$$

In conclusion, we make a few remarks regarding some possible extension of the method developed here and problems for further study. If we assume the fluid is heat-conductive, then the relation $\rho = \rho(p)$ should be replaced by the energy equation involving temperature and appropriate boundary conditions for temperature should be prescribed at the free surface and rigid boundaries. The same method can be carried through if the coefficient of heat conductivity is sufficiently large. The critical case $R = R_c$ certainly is an interesting problem and it should be of importance to derive an asymptotic equation near and at $R = R_c$ to replace the Burgers equation. Furthermore, if $R > R_c$, the Burgers equation may be ill-posed and a study of the ill-posed problem is also of significance.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER #2520	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Nonlinear Gravity-Capillary Waves on a Compressible Viscous Fluid with Edge Constraints		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) M. C. Shen		8. CONTRACT OR GRANT NUMBER(s) MCS-800-1960 DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 2 - Physical Mathematics
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		12. REPORT DATE May 1983
		13. NUMBER OF PAGES 18
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Gravity-Capillary waves, compressible viscous fluid, edge constraints, Burgers equation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An asymptotic method is developed for the study of gravity-capillary waves in a compressible viscous fluid with edge constraints in an inclined, straight channel. The Navier-Stokes equations subject to free surface and rigid bottom conditions are reduced to a sequence of elliptic boundary problems over a cross section of the channel. Their solutions are used to determine the wave speed and to construct the Burgers equation for the evolution of the gravity-capillary waves. The Burgers equation may become ill-posed when the Reynolds number exceeds some critical value. A criterion for the stability of the flow is then defined in terms of the critical Reynolds number.

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